Cambridge International A Level

Cambridge International Examinations
Cambridge International Advanced Level

FURTHER MATHEMATICS **9231/13**

Paper 1 **May/June 2015**

3 hours

Additional Materials: Answer Booklet/Paper
Graph Paper
List of Formulae (MF10)

5151870499

READ THESE INSTRUCTIONS FIRST

If you have been given an Answer Booklet, follow the instructions on the front cover of the Booklet.
Write your Centre number, candidate number and name on all the work you hand in.
Write in dark blue or black pen.
You may use an HB pencil for any diagrams or graphs.
Do not use staples, paper clips, glue or correction fluid.
DO **NOT** WRITE IN ANY BARCODES.

Answer **all** the questions.
Give non-exact numerical answers correct to 3 significant figures, or 1 decimal place in the case of angles in degrees, unless a different level of accuracy is specified in the question.
The use of a calculator is expected, where appropriate.
Results obtained solely from a graphic calculator, without supporting working or reasoning, will not receive credit.
You are reminded of the need for clear presentation in your answers.

At the end of the examination, fasten all your work securely together.
The number of marks is given in brackets [ ] at the end of each question or part question.

This document consists of **4** printed pages.

© UCLES 2015

1 The quartic equation $x^4 - px^2 + qx - r = 0$, where p, q and r are real constants, has two pairs of equal roots. Show that $p^2 + 4r = 0$ and state the value of q. [6]

2 The curve C has polar equation $r = e^{4\theta}$ for $0 \leqslant \theta \leqslant \alpha$, where α is measured in radians. The length of C is 2015. Find the value of α. [6]

3 Prove by mathematical induction that, for all positive integers n, $\displaystyle\sum_{r=1}^{n} \frac{1}{(2r)^2 - 1} = \frac{n}{2n+1}$. [6]

State the value of $\displaystyle\sum_{r=1}^{\infty} \frac{1}{(2r)^2 - 1}$. [1]

4 Use the formula for $\tan(A - B)$ in the List of Formulae (MF10) to show that

$$\tan^{-1}(x+1) - \tan^{-1}(x-1) = \tan^{-1}\left(\frac{2}{x^2}\right). \quad [3]$$

Deduce the sum to n terms of the series

$$\tan^{-1}\left(\frac{2}{1^2}\right) + \tan^{-1}\left(\frac{2}{2^2}\right) + \tan^{-1}\left(\frac{2}{3^2}\right) + \ldots. \quad [4]$$

5 Let $I_n = \displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin 2n\theta}{\cos\theta}\,\mathrm{d}\theta$, where n is a non-negative integer.

(i) Use the identity $\sin P + \sin Q \equiv 2\sin\frac{1}{2}(P+Q)\cos\frac{1}{2}(P-Q)$ to show that

$$I_n + I_{n-1} = \frac{2}{2n-1}, \text{ for all positive integers } n. \quad [5]$$

(ii) Find the exact value of $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{\sin 8\theta}{\cos\theta}\,\mathrm{d}\theta$. [4]

6 Let $z = \cos\theta + \mathrm{i}\sin\theta$. Use the binomial expansion of $(1+z)^n$, where n is a positive integer, to show that

$$\binom{n}{1}\cos\theta + \binom{n}{2}\cos 2\theta + \ldots + \binom{n}{n}\cos n\theta = 2^n\cos^n\left(\tfrac{1}{2}\theta\right)\cos\left(\tfrac{1}{2}n\theta\right) - 1. \quad [7]$$

Find

$$\binom{n}{1}\sin\theta + \binom{n}{2}\sin 2\theta + \ldots + \binom{n}{n}\sin n\theta. \quad [2]$$

7 The curve C has equation $x^2 + 2xy - 4y^2 + 20 = 0$. Show that if the tangent to C at the point (x, y) is parallel to the x-axis then $x + y = 0$. [3]

Hence find the coordinates of the stationary points on C, and determine their nature. [7]

© UCLES 2015

8 A line, passing through the point $\boldsymbol{A}(3, 0, 2)$, has vector equation $\mathbf{r} = 3\mathbf{i} + 2\mathbf{k} + \lambda(2\mathbf{i} + \mathbf{j} - 2\mathbf{k})$. It meets the plane Π, which has equation $\mathbf{r}.(\mathbf{i} + 2\mathbf{j} + \mathbf{k}) = 3$, at the point $\boldsymbol{P}$. Find the coordinates of $\boldsymbol{P}$. [3]

Write down a vector $\mathbf{n}$ which is perpendicular to Π, and calculate the vector $\mathbf{w}$, where

$$\mathbf{w} = \mathbf{n} \times (2\mathbf{i} + \mathbf{j} - 2\mathbf{k}).$$ [3]

The point $\boldsymbol{Q}$ lies in Π and is the foot of the perpendicular from $\boldsymbol{A}$ to Π. Use the vector $\mathbf{w}$ to determine an equation of the line $\boldsymbol{PQ}$ in the form $\mathbf{r} = \mathbf{u} + \mu\mathbf{v}$. [4]

9 Find the particular solution of the differential equation

$$\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}\boldsymbol{t}^2} - 3\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{t}} - 10\boldsymbol{x} = 2\sin\boldsymbol{t} - 3\cos\boldsymbol{t},$$

given that, when $\boldsymbol{t} = 0$, $\boldsymbol{x} = 3.3$ and $\dfrac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}\boldsymbol{t}} = 0.9$. [11]

10 The curve $\boldsymbol{C}$ has equation $\boldsymbol{y} = \dfrac{4\boldsymbol{x}^2 - 3\boldsymbol{x}}{\boldsymbol{x}^2 + 1}$. Verify that the equation of $\boldsymbol{C}$ may be written in the form $\boldsymbol{y} = -\dfrac{1}{2} + \dfrac{(3\boldsymbol{x} - 1)^2}{2(\boldsymbol{x}^2 + 1)}$ and also in the form $\boldsymbol{y} = \dfrac{9}{2} - \dfrac{(\boldsymbol{x} + 3)^2}{2(\boldsymbol{x}^2 + 1)}$. [3]

Hence show that $-\frac{1}{2} \leqslant \boldsymbol{y} \leqslant \frac{9}{2}$. [2]

Without differentiating, write down the coordinates of the turning points of $\boldsymbol{C}$. [2]

State the equation of the asymptote of $\boldsymbol{C}$. [1]

Sketch the graph of $\boldsymbol{C}$, stating the coordinates of the intersections with the coordinate axes and the asymptote. [3]

[Question 11 is printed on the next page.]

11 Answer only **one** of the following two alternatives.

EITHER

The linear transformation $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^4$ is represented by the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & -1 & 2 & 3 \\ 1 & -3 & 3 & 5 \\ 1 & 4 & 2 & 2 \end{pmatrix}.$$

The range space of T is denoted by $\boldsymbol{V}$.

(i) Determine the dimension of $\boldsymbol{V}$. [3]

(ii) Show that the vectors $\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1 \\ -3 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 3 \\ 2 \end{pmatrix}$ are a basis of $\boldsymbol{V}$. [5]

The set of elements of $\mathbb{R}^4$ which do not belong to $\boldsymbol{V}$ is denoted by $\boldsymbol{W}$.

(iii) State, with a reason, whether $\boldsymbol{W}$ is a vector space. [1]

(iv) Show that if the vector $\begin{pmatrix} \boldsymbol{x} \\ \boldsymbol{y} \\ z \\ \boldsymbol{t} \end{pmatrix}$ belongs to $\boldsymbol{W}$ then $\boldsymbol{x} + \boldsymbol{y} \neq z + \boldsymbol{t}$. [5]

OR

One of the eigenvalues of the matrix $\mathbf{M}$, where

$$\mathbf{M} = \begin{pmatrix} 3 & -4 & 2 \\ -4 & \alpha & 6 \\ 2 & 6 & -2 \end{pmatrix},$$

is -9. Find the value of α. [3]

Find

(i) the other two eigenvalues, λ_1 and λ_2, of $\mathbf{M}$, where $\lambda_1 > \lambda_2$, [5]

(ii) corresponding eigenvectors for all three eigenvalues of $\mathbf{M}$. [3]

It is given that $\mathbf{x} = \boldsymbol{a}\mathbf{e}_1 + \boldsymbol{b}\mathbf{e}_2$, where $\mathbf{e}_1$ and $\mathbf{e}_2$ are eigenvectors of $\mathbf{M}$ corresponding to the eigenvalues λ_1 and λ_2 respectively, and $\boldsymbol{a}$ and $\boldsymbol{b}$ are scalar constants. Show that $\mathbf{M}\mathbf{x} = \boldsymbol{p}\mathbf{e}_1 + \boldsymbol{q}\mathbf{e}_2$, expressing $\boldsymbol{p}$ and $\boldsymbol{q}$ in terms of $\boldsymbol{a}$ and $\boldsymbol{b}$. [3]

Permission to reproduce items where third-party owned material protected by copyright is included has been sought and cleared where possible. Every reasonable effort has been made by the publisher (UCLES) to trace copyright holders, but if any items requiring clearance have unwittingly been included, the publisher will be pleased to make amends at the earliest possible opportunity.

To avoid the issue of disclosure of answer-related information to candidates, all copyright acknowledgements are reproduced online in the Cambridge International Examinations Copyright Acknowledgements Booklet. This is produced for each series of examinations and is freely available to download at www.cie.org.uk after the live examination series.

Cambridge International Examinations is part of the Cambridge Assessment Group. Cambridge Assessment is the brand name of University of Cambridge Local Examinations Syndicate (UCLES), which is itself a department of the University of Cambridge.

© UCLES 2015