CAMBRIDGE INTERNATIONAL EXAMINATIONS

GCE Advanced Level

MARK SCHEME for the May/June 2013 series

9231 FURTHER MATHEMATICS

9231/21 Paper 2, maximum raw mark 100

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge will not enter into discussions about these mark schemes.

Cambridge is publishing the mark schemes for the May/June 2013 series for most IGCSE, GCE Advanced Level and Advanced Subsidiary Level components and some Ordinary Level components.

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Mark Scheme Notes

Marks are of the following three types:

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

- When a part of a question has two or more "method" steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly when there are several B marks allocated. The notation DM or DB (or dep*) is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

- The symbol √ implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only. A and B marks are not given for fortuitously "correct" answers or results obtained from incorrect working.

- Note: B2 or A2 means that the candidate can earn 2 or 0.
  B2/1/0 means that the candidate can earn anything from 0 to 2.

  The marks indicated in the scheme may not be subdivided. If there is genuine doubt whether a candidate has earned a mark, allow the candidate the benefit of the doubt. Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored.

- Wrong or missing units in an answer should not lead to the loss of a mark unless the scheme specifically indicates otherwise.

- For a numerical answer, allow the A or B mark if a value is obtained which is correct to 3 s.f., or which would be correct to 3 s.f. if rounded (1 d.p. in the case of an angle). As stated above, an A or B mark is not given if a correct numerical answer arises fortuitously from incorrect working. For Mechanics questions, allow A or B marks for correct answers which arise from taking *g* equal to 9.8 or 9.81 instead of 10.

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The following abbreviations may be used in a mark scheme or used on the scripts:

AEF Any Equivalent Form (of answer is equally acceptable)

AG Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)

BOD Benefit of Doubt (allowed when the validity of a solution may not be absolutely clear)

CAO Correct Answer Only (emphasising that no "follow through" from a previous error is allowed)

CWO Correct Working Only – often written by a “fortuitous” answer

ISW Ignore Subsequent Working

MR Misread

PA Premature Approximation (resulting in basically correct work that is insufficiently accurate)

SOS See Other Solution (the candidate makes a better attempt at the same question)

SR Special Ruling (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)

Penalties

MR –1 A penalty of MR –1 is deducted from A or B marks when the data of a question or part question are genuinely misread and the object and difficulty of the question remain unaltered. In this case all A and B marks then become "follow through √" marks. MR is not applied when the candidate misreads his own figures – this is regarded as an error in accuracy. An MR–2 penalty may be applied in particular cases if agreed at the coordination meeting.

PA –1 This is deducted from A or B marks in the case of premature approximation. The PA –1 penalty is usually discussed at the meeting.

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Question Number	Mark Scheme Details		Part Mark	Total
1	State or imply length of CP or equivalent: e.g. for angle $CPA = \theta$	$CP = 4a$ *or* $\sin\theta = 3/5$ *or* $\cos\theta = 4/5$ B1		
	EITHER: 2 moment. eqns for R and F, e.g. about P:	$5aF = 2a\,mg\cos\theta$; $F = 8mg/25$ M1;A1		
	about C:	$3aR\cos\theta - 3aF\sin\theta = a\,mg\cos\theta$ M1		
	Solve for R:	$R = (4mg + 9F)/12 = 43mg/75$ M1 A1		
	OR:	$3aT = 2a\,mg\cos\theta$ $[T = 8mg/15]$ (M1)		
	Take moments about A to give tension T:	$F = T\sin\theta$; $= 8mg/25$ (M1;A1)		
	Resolve horizontally to find friction at A:	$R = mg - T\cos\theta$; $= 43mg/75$ (M1;A1)		
	Resolve horizontally to find reaction at A:	$\mu_{\min} = 24/43$ **A.G.** M1 A1		
	Use $F = \mu_{\min} R$ to find $\mu_{\min}$:		8	**8**
2	Use conservation of momentum, e.g.:	$4mv_A + 2mv_B = 4mu$ M1		
	Use Newton's law of restitution (consistent signs)	$v_A - v_B = -eu$ M1		
	Relate v_A to u using K.E.:	$\frac{1}{2}\,4mv_A^2 = \frac{1}{4}\frac{1}{2}4mu^2$ $[v_A^2 = \frac{1}{4}u^2]$ M1		
	EITHER: Consider one possible value of v_A:	$v_A = \frac{1}{2}u$, $v_B = u$, $e = \frac{1}{2}$ B1		
	Consider other value of v_A:	$v_A = -\frac{1}{2}u$, $v_B = 3u$, $e = 3\frac{1}{2}$ (B1)		
	OR: Combine first 2 eqns to find v_A:	$v_A = \frac{1}{3}(2-e)u$, $[v_B = \frac{2}{3}(1+e)u]$		
	Find 2 possible values of e from K.E:	$(2-e)^2 = 9/4$, $e^2 - 4e + 7/4 = 0$		
		$e = \frac{1}{2}$ *or* $3\frac{1}{2}$ (B1)		
	Select one value (stating reason):	$e \le 1$ (*or* < 1) so $e = \frac{1}{2}$ **A.G.** B1	6	
	Use conservation of momentum, e.g.	$2mv_B' + mv_C = 2mv_B + \frac{1}{2}mu$ M1		
	Use Newton's law of restitution (consistent signs):	$v_B' - v_C = -e(v_B - \frac{1}{2}u)$ M1		
	Substitute $v_B = u$ and solve for v_B':	$2v_B' + v_C = 5u/2$ *and*		
		$v_B' - v_C = -\frac{1}{4}u$ so $v_B' = \frac{3}{4}u$ M1 A1		
	State why no further collisions (on v_B', v_A provided $v_B' > v_A$):	$\frac{3}{4}u > \frac{1}{2}u$ $[v_C = u$ not reqd.] B1	5	**11**

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3

Use conservation of energy: (allow B1 here if found by $v^2 = u^2 - 2gh$)
$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mga(1 - \cos\theta)$ M1 A1

Use $F = ma$ radially:
$T - mg\cos\theta = mv^2/a$ B1

Eliminate v^2 to find T:
$T = mu^2/a + mg(3\cos\theta - 2)$ **A.G.** M1 A1 5

Find or use max. and min. values of T:
$T_{max} = mu^2/a + mg$ B1
$T_{min} = mu^2/a - 5mg$ B1

Find u from $T_{max} / T_{min} = 3$:
$u^2/a + g = 3u^2/a - 15g$,
$2u^2/a = 16g$, $u = \sqrt{(8ag)}$ A.E.F. M1 A1 4

Find $\cos\theta$ from $T = \frac{1}{2}T_{max}$ [$= \frac{1}{2}\,9mg$]:
$3\cos\theta - 2 = -\frac{1}{2}\,u^2/ag + \frac{1}{2}$ M1 A1
$\cos\theta = \frac{1}{3}(2 - 4 + \frac{1}{2}) = -\frac{1}{2}$ A1 3 **12**

4

Find MI of sphere about C:
$I_{Sphere} = \frac{2}{3}\lambda m(3a)^2 + \lambda m(7a)^2$ M1 A1
Find MI of rod about C:
$I_{Rod} = \frac{1}{3}\,4m(3a/2)^2 + 4m(5a/2)^2$ M1 A1
Find MI of ring about C:
$I_{Ring} = 4m(\frac{1}{2}a)^2 + 4m(\frac{1}{2}a)^2$ M1
Combine to give MI of system about C:
$I = (55\lambda + 28 + 2)ma^2$
$= (30 + 55\lambda)ma^2$ **A.G.** A1 6

Find eqn of motion for system:
$I\,d^2\theta/dt^2 =$
$-(\lambda \times 7 + 4 \times 5/2 + 4 \times \frac{1}{2})mga\sin\theta$ M1 A1

Approximate sin θ by θ and find ω^2 in SHM eqn:
$\omega^2 = (12 + 7\lambda)g / (30 + 55\lambda)a$ M1 A1
Equate ω^2 to $4\pi^2/T^2$ to find λ:
$5(12 + 7\lambda) = 30 + 55\lambda$; $\lambda = 3/2$ M1 A1 6 **12**

5

State both hypotheses (B0 if r not ρ used):
H_0: $\rho = 0$, H_1: $\rho < 0$ B1
$r_{12,\,5\%} = 0{\cdot}497$ *B1
State or use correct tabular one-tail r value:
Accept H_0 if $|r| <$ tabular value M1
No evidence of neg. correlation A1 4 **4**
Valid method for reaching conclusion, e.g.:
Correct conclusion (AEF, dep *B1):

6

Identify distribution of X:
Negative exponential B1
State mean of X:
5/3 *or* 1·67 B1 2

(i) Find $P(X > 4)$:
$P(X > 4) = 1 - F(4)$
$= e^{-2\cdot4}$
$= 0{\cdot}0907$ M1 A1 2

(ii) State or use eqn. for median m of X:
$F(m)$ [*or* $1 - F(m)$] $= \frac{1}{2}$ M1
Find value of m:
$e^{-0\cdot6m} = \frac{1}{2}$, $m = (5/3)\ln 2$ *or* 1·16 M1 A1 3 **7**

7

Find 1st expected frequency E_2:
$E_2 = 80\int_2^3 3x^{-2}\,dx$

Find other expected frequencies:
$= 80\left[-3x^{-1}\right]_2^3$
$= 40$ **A.G.** M1 A1 4
$E_3 = 20$, $E_4 = 12$, $E_5 = 8$ M1 A1

State (at least) null hypothesis:
H_0: f(x) fits data (A.E.F.) B1
Calculate χ^2 (to 2 d.p.):
$\chi^2 = 0{\cdot}4 + 4{\cdot}05 + 0{\cdot}75 + 0{\cdot}5 = 5{\cdot}7$ M1 A1

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	State or use consistent tabular value: Conclusion (A.E.F., √ on χ^2, dep *B1):	$\chi_{3,\,0.9}{}^2 = 6{\cdot}25[1]$ $\chi^2 < 6{\cdot}25$ so f(x) does fit	*B1 B1√	5	**9**
8	*EITHER* Integrate f(x) to find F(x) for $2 \le x \le 4$: Find G(y) for $8 \le y \le 64$:	$F(x) = \int_2^x f(x)\,dx$	M1 A1		
	OR Find G(y) for $8 \le y \le 64$:	$G(y) = P(Y < y) = P(X^3 < y)$ $= P(X < y^{1/3}) = F(y^{1/3})$ $= (y^{2/3} - 4)/12$	M1 A1		
	State G(y) for other values of y: Differentiate to find g(y) for $8 \le y \le 64$:	$G(y) = P(Y < y) = P(X^3 < y)$ $= P(X < y^{1/3}) = \int_2^{y^{1/3}} f(x)\,dx$ $= \left[x^2/12\right]_2^{y^{1/3}} = \left(y^{2/3} - 4\right)/12$ $G(y) = 0\ (y < 8),\ 1\ (y > 64)$ $g(y) = y^{-1/3}/18$ **A.G.**	(M1 A1) (M1 A1) B1 B1	6	
	EITHER: Find E(Y) from $\int yg(y)$: *OR*: Find E(Y) from E(X^3): Evaluate:	$(1/18)\int_8^{64} y^{2/3}\,dy = (1/30)\left[y^{5/3}\right]_8^{64}$ $(1/6)\int_2^4 x^4\,dx = (1/30)\left[x^5\right]_8^{64}$ $= (4^5 - 2^5)/30 = 992/30$ $= 496/15$ *or* 33·1	M1 A1 (M1 A1) A1	3	**9**
9	Find sample mean and estimate population variance: (allow biased here: 1·19 *or* $1{\cdot}091^2$) State hypotheses (B0 for $\bar{x}$ …): Calculate value of t (to 3 s.f.):	$\bar{x} = 6{\cdot}1,\ s_x^2 = (384 - 61^2/10)/9$ $= 1{\cdot}322$ *or* 119/90 *or* $1{\cdot}15^2$ $H_0: \mu = 5{\cdot}2,\ H_1: \mu > 5{\cdot}2$ $t = (\bar{x} - 5{\cdot}2)/(s_x/\sqrt{10}) = 2{\cdot}47[5]$	M1 B1 M1 A1		
	State or use correct tabular t value: (or can compare $x = 6{\cdot}1$ with $5{\cdot}2 + 0{\cdot}667 = 5{\cdot}87$) Correct conclusion (A.E.F., √ on t, dep *B1):	$t_{9,\,0.95} = 1{\cdot}83[3]$ $t > 1{\cdot}83$ so mean is greater	*B1 B1√	6	
	State hypotheses (B0 for $\bar{x}$ …), e.g.: State assumption (A.E.F.): Find sample mean [and estimate variance] for Y:	$H_0: \mu_P = \mu_Q,\ H_1: \mu_P < \mu_Q$ Distributions have equal variances $y = 7{\cdot}0,\ [s_y^2 = (500{\cdot}6 - 70^2/10)/9$ $= 1{\cdot}178$ *or* 53/45 *or* $1{\cdot}085^2$]	B1 B1		
	EITHER Estimate (pooled) common varia	$s^2 = (9 \times 1{\cdot}322 + 9 \times 1{\cdot}178)/18$ *or* $(384 - 61^2/10 + 500{\cdot}6 - 70^2/10)/18$ $= 1.25$ *or* 1.18^2	M1 A1		
	OR (Equivalently) estimate common variance: Calculate value of t (to 3 s.f.):	$s^2 = (1{\cdot}322 + 1{\cdot}178)/10 = 0{\cdot}25$ $t = (\bar{y} - \bar{x})/s = 1{\cdot}8$	(M1 A1) (M1 A1)		
	State or use correct tabular t value: (or can compare $\bar{y} - \bar{x} = 0{\cdot}9$ with 0·867) Correct conclusion (A.E.F., √ on t, dep *B1):	$t_{18,\,0.95} = 1{\cdot}73[4]$ $t > 1{\cdot}73$ so Q's mean is greater	*B1 B1√	8	**14**

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10 (a)	Find K.E. after falling ka from P.E. and E.P.E.:	$mg \times ka - \frac{1}{2}(3mg/2)(ka-a)^2/a$ M1 =[A1 $= mga(k - 3k^2/4 + 3k/2 - \frac{3}{4})$ $= \frac{1}{4}mga(10k - 3 - 3k^2)$ **A.G.** A1	3	
	Find eqn. for k (*or* ka) at B using K.E. = 0:	$3k^2 - 10k + 3 = 0$ (A.E.F.) M1		
	Find both roots and select root > 1:	Roots 3 and ⅓, so $k = 3$ **A.G.** M1 A1	3	
	Find time t_1 to fall a from A (under no tension):	$a = \frac{1}{2} g t_1^2$, $t_1 = \sqrt{(2a/g)}$ B1		
	Find ext. e or distance fallen at equilibrium pt O:	$mg = 3mge/2a$, $e = 2a/3$ *or* $AO = 5a/3$ B1		
	State or find SHM eqn. at x below (or above) O:	$m\mathrm{d}^2x/\mathrm{d}t^2 = mg - 3mg(x+e)/2a$ M1 $\mathrm{d}^2x/\mathrm{d}t^2 = -3gx/2a$ A1		
	State or use correct amplitude x_0 and ω^2:	$x_0 = 4a/3$ *and* $\omega^2 = 3g/2a$ B1		
	Find time t_2 to fall from a to $3a$ below A, e.g.:	[–] $e = x_0 \cos \omega t_2$ M1 $t_2 = \sqrt{(2a/3g)} \cos^{-1}(-\frac{1}{2})$ (A.E.F.) A1		
	Find total time $t_1 + t_2$:	$\sqrt{(2a/g)} + (2\pi/3)\sqrt{(2a/3g)}$ **A.G.** A1	8	**14**
(b)	Find 4 summations reqd. in this part:	$\Sigma x = 12 + p$, $\Sigma x^2 = 38 + p^2$, $\Sigma y = 23$, [$\Sigma y^2 = 125$] $\Sigma xy = 63 + 2p$ M1		
	EITHER: Substitute $b_1 = 1$ in formula for gradient: (A.E.F., A1 for each side)	$63 + 2p - (12+p) \times 23/5$ $= 38 + p^2 - (12+p)^2/5$ *or* $39 - 13p = 4p^2 - 24p + 46$ M1 A1 A1		
	OR Substitute in normal eqns. or equivalent: (A.E.F., A1 for each eqn.)	$5k + (12 + p) = 23$ *and* $(12 + p)k + (38 + p^2) = 63 + 2p$ (M1 A1 A1)		
	Obtain and solve quadratic for p (A.E.F):	$0{\cdot}8p^2 - 2{\cdot}2p + 1{\cdot}4 = 0$ *or* $4p^2 - 11p + 7 = 0$ M1 A1 $p = 1$, $1{\cdot}75$ (*or* 7/4) A1, A1	8	

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(i)	Find eqn, for correlation coefficient with $p = 1$:				
	EITHER:		$r = (65 - 13 \times 23/5) / \sqrt{\{(39 - 13^2/5)(125 - 23^2/5)\}}$ M1 A1 $= 5{\cdot}2 / \sqrt{(5{\cdot}2 \times 19{\cdot}2)}$ *or* $1{\cdot}04 / \sqrt{(1{\cdot}04 \times 3{\cdot}84)}$		
	OR		$r^2 = 1 \times (65 - 13 \times 23/5) / (125 - 23^2/5)$ *or* $1 \times (39 - 13^2/5) / (125 - 23^2/5)$ $= 5{\cdot}2 / 19{\cdot}2$ *or* $1{\cdot}04 / 3{\cdot}84$ M1 A1		
	Evaluate r		$r = \sqrt{39/12}$ *or* $0{\cdot}52$ A1	3	
(ii)	*EITHER*:	Recall or find gradient b_2 of line:	$b_2 = 5{\cdot}2/19{\cdot}2$ *or* r^2 [= 0·2708] M1		
		Find regression line of x on y:	$x - 13/5 = b_2 (y - 23/5)$ M1 $x = (13/48)y + 65/48$ *or* $0{\cdot}271y + 1{\cdot}35$ A1		
	OR	Use normal eqns for $x = a_2 + b_2 y$:	$5a_2 + 23b_2 = 13$ *and* (M1) $23a_2 + 125b_2 = 65$ (M1)		
		Solve for a_2, b_2:	$b_2 = 13/48$ *or* 0·271 *And* $a_2 = 65/48$ *or* 1·35 (A1)	3	**14**

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